Similarly

$$(z - e^{\frac{4\pi i}{5}})(z - e^{\frac{-4\pi i}{5}}) = z^2 - 2z \cos \tfrac{4\pi}{5} + 1.$$

This gives the desired factorization.

EXAMPLE 5.7.2 Solve $z^3 = i$.

Solution. $|i| = 1$ and $\operatorname{Arg} i = \frac{\pi}{2} = \alpha$. So by equation 5.4, the solutions are

$$z_k = |i|^{1/3} e^{\frac{i(\alpha + 2k\pi)}{3}},\ k = 0,\ 1,\ 2.$$

First, $k = 0$ gives

$$z_0 = e^{\frac{i\pi}{6}} = \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{i}{2}.$$

Next, $k = 1$ gives

$$z_1 = e^{\frac{5\pi i}{6}} = \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} = \frac{-\sqrt{3}}{2} + \frac{i}{2}.$$

Finally, $k = 2$ gives

$$z_1 = e^{\frac{9\pi i}{6}} = \cos \frac{9\pi}{6} + i \sin \frac{9\pi}{6} = -i.$$

We finish this chapter with two more examples of De Moivre's theorem.

EXAMPLE 5.7.3 If

$$\begin{aligned} C &= 1 + \cos\theta + \cdots + \cos\,(n-1)\theta, \\ S &= \sin\theta + \cdots + \sin\,(n-1)\theta, \end{aligned}$$

prove that

$$C = \frac{\sin \frac{n\theta}{2}}{\sin \frac{\theta}{2}} \cos \tfrac{(n-1)\theta}{2} \text{ and } S = \frac{\sin \frac{n\theta}{2}}{\sin \frac{\theta}{2}} \sin \tfrac{(n-1)\theta}{2},$$

if $\theta \neq 2k\pi,\ k \in \mathbb{Z}$.

Solution.

$$\begin{aligned}
C+iS &= 1+(\cos\theta+i\sin\theta)+\cdots+(\cos\,(n-1)\theta+i\sin\,(n-1)\theta)\\
&= 1+e^{i\theta}+\cdots+e^{i(n-1)\theta}\\
&= 1+z+\cdots+z^{n-1},\ \text{where } z=e^{i\theta}\\
&= \frac{1-z^n}{1-z},\ \text{if } z\neq 1,\ \text{i.e. } \theta\neq 2k\pi,\\
&= \frac{1-e^{in\theta}}{1-e^{i\theta}}=\frac{e^{\frac{in\theta}{2}}(e^{\frac{-in\theta}{2}}-e^{\frac{in\theta}{2}})}{e^{\frac{i\theta}{2}}(e^{\frac{-i\theta}{2}}-e^{\frac{i\theta}{2}})}\\
&= e^{i(n-1)\frac{\theta}{2}}\frac{\sin\,\frac{n\theta}{2}}{\sin\,\frac{\theta}{2}}\\
&= (\cos\,(n-1)\tfrac{\theta}{2}+i\sin\,(n-1)\tfrac{\theta}{2})\frac{\sin\,\frac{n\theta}{2}}{\sin\,\frac{\theta}{2}}.
\end{aligned}$$

The result follows by equating real and imaginary parts.

EXAMPLE 5.7.4 Express $\cos\,n\theta$ and $\sin\,n\theta$ in terms of $\cos\theta$ and $\sin\theta$, using the equation $\cos\,n\theta+\sin\,n\theta=(\cos\theta+i\sin\theta)^n$.

Solution. The binomial theorem gives

$$(\cos\theta+i\sin\theta)^n=\cos^n\theta+\tbinom{n}{1}\cos^{n-1}\theta(i\sin\theta)+\tbinom{n}{2}\cos^{n-2}\theta(i\sin\theta)^2+\cdots + (i\sin\theta)^n.$$

Equating real and imaginary parts gives

$$\begin{aligned}
\cos\,n\theta &= \cos^n\theta-\tbinom{n}{2}\cos^{n-2}\theta\sin^2\theta+\cdots\\
\sin\,n\theta &= \tbinom{n}{1}\cos^{n-1}\theta\sin\theta-\tbinom{n}{3}\cos^{n-3}\theta\sin^3\theta+\cdots\,.
\end{aligned}$$

5.8 PROBLEMS

1. Express the following complex numbers in the form $x+iy$, x,y real:

   (i) $(-3+i)(14-2i)$; (ii) $\dfrac{2+3i}{1-4i}$; (iii) $\dfrac{(1+2i)^2}{1-i}$.

   [Answers: (i) $-40+20i$; (ii) $-\frac{10}{17}+\frac{11}{17}i$; (iii) $-\frac{7}{2}+\frac{i}{2}$.]

2. Solve the following equations:

$$\text{(i)} \qquad iz + (2-10i)z = 3z + 2i,$$

$$\text{(ii)} \qquad \begin{aligned} (1+i)z + (2-i)w &= -3i \\ (1+2i)z + (3+i)w &= 2+2i. \end{aligned}$$

[Answers:(i) $z = -\frac{9}{41} - \frac{i}{41}$; (ii) $z = -1+5i$, $w = \frac{19}{5} - \frac{8i}{5}$.]

3. Express $1 + (1+i) + (1+i)^2 + \ldots + (1+i)^{99}$ in the form $x + iy$, x, y real. [Answer: $(1+2^{50})i$.]

4. Solve the equations: (i) $z^2 = -8-6i$; (ii) $z^2 - (3+i)z + 4 + 3i = 0$.

   [Answers: (i) $z = \pm(1-3i)$; (ii) $z = 2-i,\ 1+2i$.]

5. Find the modulus and principal argument of each of the following complex numbers:

   (i) $4+i$; (ii) $-\frac{3}{2} - \frac{i}{2}$; (iii) $-1+2i$; (iv) $\frac{1}{2}(-1+i\sqrt{3})$.

   [Answers: (i) $\sqrt{17}$, $\tan^{-1}\frac{1}{4}$; (ii) $\frac{\sqrt{10}}{2}$, $-\pi + \tan^{-1}\frac{1}{3}$; (iii) $\sqrt{5}$, $\pi - \tan^{-1} 2$.]

6. Express the following complex numbers in modulus-argument form:

   (i) $z = (1+i)(1+i\sqrt{3})(\sqrt{3}-i)$.

   (ii) $z = \dfrac{(1+i)^5(1-i\sqrt{3})^5}{(\sqrt{3}+i)^4}$.

   [Answers:
   (i) $z = 4\sqrt{2}(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12})$; (ii) $z = 2^{7/2}(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12})$.]

7. (i) If $z = 2(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$ and $w = 3(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$, find the polar form of
   (a) zw; (b) $\frac{z}{w}$; (c) $\frac{w}{z}$; (d) $\frac{z^5}{w^2}$.

   (ii) Express the following complex numbers in the form $x + iy$:
   (a) $(1+i)^{12}$; (b) $\left(\frac{1-i}{\sqrt{2}}\right)^{-6}$.

   [Answers: (i): (a) $6(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12})$; (b) $\frac{2}{3}(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12})$;

   (c) $\frac{3}{2}(\cos -\frac{\pi}{12} + i\sin -\frac{\pi}{12})$; (d) $\frac{32}{9}(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12})$;

   (ii): (a) -64; (b) $-i$.]

8. Solve the equations:

$$\text{(i) } z^2 = 1 + i\sqrt{3};\ \text{(ii) } z^4 = i;\ \text{(iii) } z^3 = -8i;\ \text{(iv) } z^4 = 2 - 2i.$$

[Answers: (i) $z = \pm\frac{(\sqrt{3}+i)}{\sqrt{2}}$; (ii) $i^k(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}), k = 0, 1, 2, 3$; (iii) $z = 2i,\ -\sqrt{3} - i,\ \sqrt{3} - i$; (iv) $z = i^k 2^{\frac{3}{8}}(\cos\frac{\pi}{16} - i\sin\frac{\pi}{16}),\ k = 0, 1, 2, 3$.]

9. Find the reduced row–echelon form of the complex matrix

$$\begin{bmatrix} 2+i & -1+2i & 2 \\ 1+i & -1+i & 1 \\ 1+2i & -2+i & 1+i \end{bmatrix}.$$

[Answer: $\begin{bmatrix} 1 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.]

10. (i) Prove that the line equation $lx + my = n$ is equivalent to

$$\overline{p}z + p\overline{z} = 2n,$$

where $p = l + im$.

(ii) Use (ii) to deduce that reflection in the straight line

$$\overline{p}z + p\overline{z} = n$$

is described by the equation

$$\overline{p}w + p\overline{z} = n.$$

[Hint: The complex number $l + im$ is perpendicular to the given line.]

(iii) Prove that the line $|z-a| = |z-b|$ may be written as $\overline{p}z + p\overline{z} = n$, where $p = b - a$ and $n = |b|^2 - |a|^2$. Deduce that if z lies on the Apollonius circle $\frac{|z-a|}{|z-b|} = \lambda$, then w, the reflection of z in the line $|z-a| = |z-b|$, lies on the Apollonius circle $\frac{|z-a|}{|z-b|} = \frac{1}{\lambda}$.

11. Let a and b be distinct complex numbers and $0 < \alpha < \pi$.

(i) Prove that each of the following sets in the complex plane represents a circular arc and sketch the circular arcs on the same diagram:

$$\operatorname{Arg} \frac{z-a}{z-b} = \alpha,\ -\alpha,\ \pi-\alpha,\ \alpha-\pi.$$

Also show that $\operatorname{Arg} \frac{z-a}{z-b} = \pi$ represents the line segment joining a and b, while $\operatorname{Arg} \frac{z-a}{z-b} = 0$ represents the remaining portion of the line through a and b.

(ii) Use (i) to prove that four distinct points z_1, z_2, z_3, z_4 are concyclic or collinear, if and only if the *cross–ratio*

$$\frac{z_4-z_1}{z_4-z_2} \Big/ \frac{z_3-z_1}{z_3-z_2}$$

is real.

(iii) Use (ii) to derive *Ptolemy's* Theorem: Four distinct points A, B, C, D are concyclic or collinear, if and only if one of the following holds:

$$\begin{aligned} AB\cdot CD + BC\cdot AD &= AC\cdot BD \\ BD\cdot AC + AD\cdot BC &= AB\cdot CD \\ BD\cdot AC + AB\cdot CD &= AD\cdot BC. \end{aligned}$$

Chapter 6

EIGENVALUES AND EIGENVECTORS

6.1 Motivation

We motivate the chapter on eigenvalues by discussing the equation

$$ax^2 + 2hxy + by^2 = c,$$

where not all of a, h, b are zero. The expression $ax^2 + 2hxy + by^2$ is called a *quadratic form* in x and y and we have the identity

$$ax^2 + 2hxy + by^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = X^t AX,$$

where $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$. A is called the matrix of the quadratic form.

We now rotate the x, y axes anticlockwise through θ radians to new x_1, y_1 axes. The equations describing the rotation of axes are derived as follows:

Let P have coordinates (x, y) relative to the x, y axes and coordinates (x_1, y_1) relative to the x_1, y_1 axes. Then referring to Figure 6.1:

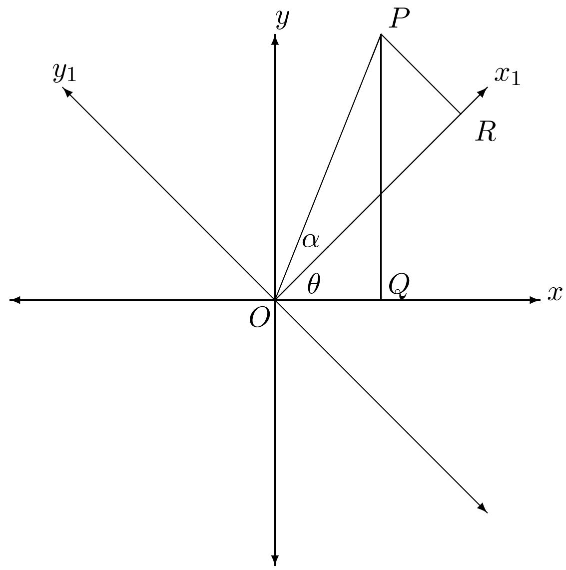

Figure 6.1: Rotating the axes.

$$
\begin{aligned}
x &= OQ = OP \cos\,(\theta+\alpha) \\
&= OP(\cos\theta\cos\alpha - \sin\theta\sin\alpha) \\
&= (OP\cos\alpha)\cos\theta - (OP\sin\alpha)\sin\theta \\
&= OR\cos\theta - PR\sin\theta \\
&= x_1\cos\theta - y_1\sin\theta.
\end{aligned}
$$

Similarly $y = x_1\sin\theta + y_1\cos\theta$.

We can combine these transformation equations into the single matrix equation:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix},$$

or $X = PY$, where $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $Y = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. We note that the columns of P give the directions of the positive x_1 and y_1 axes. Also P is an orthogonal matrix – we have $PP^t = I_2$ and so $P^{-1} = P^t$. The matrix P has the special property that $\det P = 1$.

A matrix of the type $P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is called a *rotation* matrix. We shall show soon that any 2×2 real orthogonal matrix with determinant equal to 1 is a rotation matrix.

We can also solve for the new coordinates in terms of the old ones:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = Y = P^t X = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

so $x_1 = x\cos\theta + y\sin\theta$ and $y_1 = -x\sin\theta + y\cos\theta$. Then

$$X^t AX = (PY)^t A(PY) = Y^t(P^t AP)Y.$$

Now suppose, as we later show, that it is possible to choose an angle θ so that $P^t AP$ is a diagonal matrix, say $\mathrm{diag}(\lambda_1,\, \lambda_2)$. Then

$$X^t AX = \begin{bmatrix} x_1 & y_1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \lambda_1 x_1^2 + \lambda_2 y_1^2 \tag{6.1}$$

and relative to the new axes, the equation $ax^2 + 2hxy + by^2 = c$ becomes $\lambda_1 x_1^2 + \lambda_2 y_1^2 = c$, which is quite easy to sketch. This curve is symmetrical about the x_1 and y_1 axes, with P_1 and P_2, the respective columns of P, giving the directions of the axes of symmetry.

Also it can be verified that P_1 and P_2 satisfy the equations

$$AP_1 = \lambda_1 P_1 \text{ and } AP_2 = \lambda_2 P_2.$$

These equations force a restriction on λ_1 and λ_2. For if $P_1 = \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}$, the first equation becomes

$$\begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \lambda_1 \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} \text{ or } \begin{bmatrix} a-\lambda_1 & h \\ h & b-\lambda_1 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Hence we are dealing with a homogeneous system of two linear equations in two unknowns, having a non–trivial solution $(u_1,\, v_1)$. Hence

$$\begin{vmatrix} a-\lambda_1 & h \\ h & b-\lambda_1 \end{vmatrix} = 0.$$

Similarly, λ_2 satisfies the same equation. In expanded form, λ_1 and λ_2 satisfy

$$\lambda^2 - (a+b)\lambda + ab - h^2 = 0.$$

This equation has real roots

$$\lambda = \frac{a+b \pm \sqrt{(a+b)^2 - 4(ab-h^2)}}{2} = \frac{a+b \pm \sqrt{(a-b)^2 + 4h^2}}{2} \tag{6.2}$$

(The roots are distinct if $a \neq b$ or $h \neq 0$. The case $a = b$ and $h = 0$ needs no investigation, as it gives an equation of a circle.)

The equation $\lambda^2 - (a+b)\lambda + ab - h^2 = 0$ is called the *eigenvalue equation* of the matrix A.

6.2 Definitions and examples

DEFINITION 6.2.1 (Eigenvalue, eigenvector)
Let A be a complex square matrix. Then if λ is a complex number and X a *non–zero* complex column vector satisfying $AX = \lambda X$, we call X an *eigenvector* of A, while λ is called an *eigenvalue* of A. We also say that X is an eigenvector corresponding to the eigenvalue λ.

So in the above example P_1 and P_2 are eigenvectors corresponding to λ_1 and λ_2, respectively. We shall give an algorithm which starts from the eigenvalues of $A = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$ and constructs a rotation matrix P such that P^tAP is diagonal.

As noted above, if λ is an eigenvalue of an $n \times n$ matrix A, with corresponding eigenvector X, then $(A - \lambda I_n)X = 0$, with $X \neq 0$, so $\det(A - \lambda I_n) = 0$ and there are at most n distinct eigenvalues of A.

Conversely if $\det(A - \lambda I_n) = 0$, then $(A - \lambda I_n)X = 0$ has a non–trivial solution X and so λ is an eigenvalue of A with X a corresponding eigenvector.

DEFINITION 6.2.2 (Characteristic equation, polynomial)
The equation $\det(A - \lambda I_n) = 0$ is called the *characteristic equation* of A, while the polynomial $\det(A - \lambda I_n)$ is called the *characteristic polynomial* of A. The characteristic polynomial of A is often denoted by $\mathrm{ch}_A(\lambda)$.

Hence the eigenvalues of A are the roots of the characteristic polynomial of A.

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, it is easily verified that the characteristic polynomial is $\lambda^2 - (\mathrm{trace}\, A)\lambda + \det A$, where $\mathrm{trace}\, A = a + d$ is the sum of the diagonal elements of A.

EXAMPLE 6.2.1 Find the eigenvalues of $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and find all eigenvectors.

Solution. The characteristic equation of A is $\lambda^2 - 4\lambda + 3 = 0$, or

$$(\lambda - 1)(\lambda - 3) = 0.$$

Hence $\lambda = 1$ or 3. The eigenvector equation $(A - \lambda I_n)X = 0$ reduces to

$$\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

or

$$\begin{aligned}(2-\lambda)x + y &= 0\\ x + (2-\lambda)y &= 0.\end{aligned}$$

Taking $\lambda = 1$ gives

$$\begin{aligned}x + y &= 0\\ x + y &= 0,\end{aligned}$$

which has solution $x = -y$, y arbitrary. Consequently the eigenvectors corresponding to $\lambda = 1$ are the vectors $\begin{bmatrix} -y \\ y \end{bmatrix}$, with $y \neq 0$.

Taking $\lambda = 3$ gives

$$\begin{aligned}-x + y &= 0\\ x - y &= 0,\end{aligned}$$

which has solution $x = y$, y arbitrary. Consequently the eigenvectors corresponding to $\lambda = 3$ are the vectors $\begin{bmatrix} y \\ y \end{bmatrix}$, with $y \neq 0$.

Our next result has wide applicability:

THEOREM 6.2.1 Let A be a 2×2 matrix having distinct eigenvalues λ_1 and λ_2 and corresponding eigenvectors X_1 and X_2. Let P be the matrix whose columns are X_1 and X_2, respectively. Then P is non–singular and

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Proof. Suppose $AX_1 = \lambda_1 X_1$ and $AX_2 = \lambda_2 X_2$. We show that the system of homogeneous equations

$$xX_1 + yX_2 = 0$$

has only the trivial solution. Then by theorem 2.5.10 the matrix $P = [X_1|X_2]$ is non–singular. So assume

$$xX_1 + yX_2 = 0. \tag{6.3}$$

Then $A(xX_1 + yX_2) = A0 = 0$, so $x(AX_1) + y(AX_2) = 0$. Hence

$$x\lambda_1 X_1 + y\lambda_2 X_2 = 0. \tag{6.4}$$

Multiplying equation 6.3 by λ_1 and subtracting from equation 6.4 gives

$$(\lambda_2 - \lambda_1)yX_2 = 0.$$

Hence $y = 0$, as $(\lambda_2 - \lambda_1) \neq 0$ and $X_2 \neq 0$. Then from equation 6.3, $xX_1 = 0$ and hence $x = 0$.

Then the equations $AX_1 = \lambda_1 X_1$ and $AX_2 = \lambda_2 X_2$ give

$$\begin{aligned} AP = A[X_1|X_2] = [AX_1|AX_2] &= [\lambda_1 X_1|\lambda_2 X_2] \\ &= [X_1|X_2]\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = P\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \end{aligned}$$

so

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

EXAMPLE 6.2.2 Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ be the matrix of example 6.2.1. Then $X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are eigenvectors corresponding to eigenvalues 1 and 3, respectively. Hence if $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, we have

$$P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

There are two immediate applications of theorem 6.2.1. The first is to the calculation of A^n: If $P^{-1}AP = \mathrm{diag}\,(\lambda_1, \lambda_2)$, then $A = P\mathrm{diag}\,(\lambda_1, \lambda_2)P^{-1}$ and

$$A^n = \left(P\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}P^{-1}\right)^n = P\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^n P^{-1} = P\begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}P^{-1}.$$

The second application is to solving a system of linear differential equations

$$\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy, \end{aligned}$$

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a matrix of real or complex numbers and x and y are functions of t. The system can be written in matrix form as $\dot{X} = AX$, where

$$X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } \dot{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}.$$

We make the substitution $X = PY$, where $Y = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$. Then x_1 and y_1 are also functions of t and

$$\dot{X} = P\dot{Y} = AX = A(PY), \text{ so } \dot{Y} = (P^{-1}AP)Y = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} Y.$$

Hence $\dot{x_1} = \lambda_1 x_1$ and $\dot{y_1} = \lambda_2 y_1$.

These differential equations are well–known to have the solutions $x_1 = x_1(0)e^{\lambda_1 t}$ and $x_2 = x_2(0)e^{\lambda_2 t}$, where $x_1(0)$ is the value of x_1 when $t = 0$.

[If $\frac{dx}{dt} = kx$, where k is a constant, then

$$\frac{d}{dt}\left(e^{-kt}x\right) = -ke^{-kt}x + e^{-kt}\frac{dx}{dt} = -ke^{-kt}x + e^{-kt}kx = 0.$$

Hence $e^{-kt}x$ is constant, so $e^{-kt}x = e^{-k0}x(0) = x(0)$. Hence $x = x(0)e^{kt}$.]

However $\begin{bmatrix} x_1(0) \\ y_1(0) \end{bmatrix} = P^{-1}\begin{bmatrix} x(0) \\ y(0) \end{bmatrix}$, so this determines $x_1(0)$ and $y_1(0)$ in terms of $x(0)$ and $y(0)$. Hence ultimately x and y are determined as explicit functions of t, using the equation $X = PY$.

EXAMPLE 6.2.3 Let $A = \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix}$. Use the eigenvalue method to derive an explicit formula for A^n and also solve the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= 2x - 3y \\ \frac{dy}{dt} &= 4x - 5y, \end{aligned}$$

given $x = 7$ and $y = 13$ when $t = 0$.

Solution. The characteristic polynomial of A is $\lambda^2+3\lambda+2$ which has distinct roots $\lambda_1 = -1$ and $\lambda_2 = -2$. We find corresponding eigenvectors $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Hence if $P = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$, we have $P^{-1}AP = \text{diag}\,(-1, -2)$. Hence

$$\begin{aligned} A^n &= \left(P\text{diag}\,(-1, -2)P^{-1}\right)^n = P\text{diag}\,((-1)^n, (-2)^n)P^{-1} \\ &= \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} (-1)^n & 0 \\ 0 & (-2)^n \end{bmatrix}\begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
&= (-1)^n \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2^n \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \\
&= (-1)^n \begin{bmatrix} 1 & 3 \times 2^n \\ 1 & 4 \times 2^n \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \\
&= (-1)^n \begin{bmatrix} 4 - 3 \times 2^n & -3 + 3 \times 2^n \\ 4 - 4 \times 2^n & -3 + 4 \times 2^n \end{bmatrix}.
\end{aligned}
$$

To solve the differential equation system, make the substitution $X = PY$. Then $x = x_1 + 3y_1$, $y = x_1 + 4y_1$. The system then becomes

$$
\begin{aligned}
\dot{x}_1 &= -x_1 \\
\dot{y}_1 &= -2y_1,
\end{aligned}
$$

so $x_1 = x_1(0)e^{-t}$, $y_1 = y_1(0)e^{-2t}$. Now

$$
\begin{bmatrix} x_1(0) \\ y_1(0) \end{bmatrix} = P^{-1} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 13 \end{bmatrix} = \begin{bmatrix} -11 \\ 6 \end{bmatrix},
$$

so $x_1 = -11e^{-t}$ and $y_1 = 6e^{-2t}$. Hence $x = -11e^{-t} + 3(6e^{-2t}) = -11e^{-t} + 18e^{-2t}$, $y = -11e^{-t} + 4(6e^{-2t}) = -11e^{-t} + 24e^{-2t}$.

For a more complicated example we solve a system of *inhomogeneous* recurrence relations.

EXAMPLE 6.2.4 Solve the system of recurrence relations

$$
\begin{aligned}
x_{n+1} &= 2x_n - y_n - 1 \\
y_{n+1} &= -x_n + 2y_n + 2,
\end{aligned}
$$

given that $x_0 = 0$ and $y_0 = -1$.

Solution. The system can be written in matrix form as

$$X_{n+1} = AX_n + B,$$

where

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

It is then an easy induction to prove that

$$X_n = A^n X_0 + (A^{n-1} + \cdots + A + I_2)B. \tag{6.5}$$

Also it is easy to verify by the eigenvalue method that

$$A^n = \frac{1}{2}\begin{bmatrix} 1+3^n & 1-3^n \\ 1-3^n & 1+3^n \end{bmatrix} = \frac{1}{2}U + \frac{3^n}{2}V,$$

where $U = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $V = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Hence

$$\begin{aligned} A^{n-1} + \cdots + A + I_2 &= \frac{n}{2}U + \frac{(3^{n-1} + \cdots + 3 + 1)}{2}V \\ &= \frac{n}{2}U + \frac{(3^{n-1} - 1)}{4}V. \end{aligned}$$

Then equation 6.5 gives

$$X_n = \left(\frac{1}{2}U + \frac{3^n}{2}V\right)\begin{bmatrix} 0 \\ -1 \end{bmatrix} + \left(\frac{n}{2}U + \frac{(3^{n-1}-1)}{4}V\right)\begin{bmatrix} -1 \\ 2 \end{bmatrix},$$

which simplifies to

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} (2n+1-3^n)/4 \\ (2n-5+3^n)/4 \end{bmatrix}.$$

Hence $x_n = (2n - 1 + 3^n)/4$ and $y_n = (2n - 5 + 3^n)/4$.

REMARK 6.2.1 If $(A - I_2)^{-1}$ existed (that is, if $\det(A - I_2) \neq 0$, or equivalently, if 1 is not an eigenvalue of A), then we could have used the formula

$$A^{n-1} + \cdots + A + I_2 = (A^n - I_2)(A - I_2)^{-1}. \tag{6.6}$$

However the eigenvalues of A are 1 and 3 in the above problem, so formula 6.6 cannot be used there.

Our discussion of eigenvalues and eigenvectors has been limited to 2×2 matrices. The discussion is a more complicated for matrices of size greater than two and is best left to a second course in linear algebra. Nevertheless the following result is a useful generalization of theorem 6.2.1. The reader is referred to [28, page 350] for a proof.

THEOREM 6.2.2 Let A be an $n \times n$ matrix having distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $X_1, \ldots, X_n$. Let P be the matrix whose columns are respectively $X_1, \ldots, X_n$. Then P is non–singular and

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$